NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL NOTE 3559

LAMINAR SEPARATION OVER A TRANSPIRATION-COOLED SURFACE IN COMPRESSIBLE FLOW

By Morris Morduchow

Polytechnic Institute of Brooklyn

Washington
December 1955

NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL NOTE 3559

LAMINAR SEPARATION OVER A TRANSPIRATION-COOLED SURFACE IN COMPRESSIBLE FLOW

By Morris Morduchow

SUMMARY

A theoretical analysis of laminar separation in compressible flow over a transpiration-cooled surface maintained at a uniform wall temperature is made. A simple method of calculating the separation point over such a surface for a given adverse pressure gradient, Mach number, wall temperature, and uniform coolant temperature is developed. This method is expected to be sufficiently accurate for most practical purposes. To show the effects of these parameters on the separation point a numerical example is worked out in detail. The normal mass flow is found to have a predominating effect on the location of the separation point, since over a transpiration-cooled wall separation is found to occur upstream of the separation point over a heat-insulated wall without normal mass flow at the same adverse pressure gradient and Mach number. The method of analysis is based on an application of the Kármán momentum integral equation in conjunction with seventh-degree velocity and stagnation-enthalpy profiles.

INTRODUCTION

The cooling of aerodynamically heated surfaces has recently gained considerable attention in connection with rocket walls, turbine blades, and high-speed flow over a wing. A promising means of cooling such a surface appears to be that of transpiration or sweat cooling, a method by which the surface is made porous and a comparatively small quantity, per unit time, of cool fluid is injected normally through the pores into the main stream (ref. 1).

The question of separation may be of particular interest in connection with flow over a transpiration-cooled surface since a normal mass flow strongly tends to promote separation by moving the separation point upstream. On the other hand, cooling of the wall tends, by itself, to delay separation by moving the separation point downstream.

The purpose of this study is to determine theoretically the actual net effect of simultaneous normal mass flow and cooling of the wall on conditions of separation over a sweat-cooled surface. Separation, as is well known, is a condition to be avoided, especially in order to prevent high drag.

The present analysis is based on a study of laminar flow in the boundary layer. It is true that in many actual cases of sweat cooling, such as the cooling of turbine blades, the flow is perhaps more likely to be turbulent than laminar. However, the present state of basic theoretical knowledge on turbulent flow, especially turbulent separation, does not appear adequate to serve in developing a satisfactory analysis of separation over a sweat-cooled wall. For laminar flow, on the other hand, the existing basic theoretical knowledge is, in a sense, complete (if the Navier-Stokes equations, together with Prandtl's boundary-layer simplifications, are accepted), and it will be seen that it is possible, on this basis, to develop a fairly simple and yet sufficiently accurate means of analyzing laminar separation over a sweat-cooled wall. Since separation tends to occur more readily in a laminar than in a turbulent boundary layer, the present study is of interest in itself. Moreover, it may tentatively serve as a possible qualitative guide to the corresponding effects of parameters such as the mass-flow injection-ratio or coolant-temperature parameter, the wall-temperature ratio, and the Mach number on separation in turbulent flow.

Although laminar separation over an impermeable wall has received considerable attention in the literature (refs. 2 to 6), the only analysis of separation over a transpiration-cooled wall appears to be that in reference 7, where an application of the Kármán-Pohlhausen method, based on fourth-degree velocity profiles, was made. Because of the appreciable quantitative inaccuracies inherent in the use of fourth-degree profiles for determining the separation point, however, the analysis of separation in reference 7 must be regarded as only qualitative. The present analysis, which may be considered as a refinement of that in reference 7, is also based on the Kármán momentum integral equation but in conjunction with seventh-degree velocity profiles, as in references 2 and 6. By using seventh- instead of fourth-degree velocity profiles, it is possible to satisfy additional boundary conditions (especially at the wall) which an exact solution would necessarily satisfy. As shown in references 2 and 6, excellent agreement, even at high Mach numbers, with the zero-heat-transfer results of reference 8 has thereby been obtained.

Although the present analysis can be readily used as a basis for investigating other properties of the laminar boundary layer over a transpiration-cooled wall, the emphasis here is on only one aspect, separation. A simple method of calculating other properties (such as heat transfer and skin friction) of the laminar boundary layer in flow over a sweat-cooled surface maintained at a uniform wall temperature is given in reference 9. A method of calculating the required normal-mass-flow injection distribution is also given in reference 9. Like reference 7, however, reference 9 is based on the use of fourth-degree velocity profiles and cannot therefore lead to quantitatively accurate results for the location of the separation point in an adverse pressure gradient.

In the present investigation, the wall is assumed to be maintained at a uniform temperature by the appropriate distribution of normal mass flow of the coolant fluid. The coolant is assumed to be the same fluid as that of the main stream (namely, air), while the coolant temperature is assumed to be uniform along the surface.

It is further assumed, for mathematical simplicity, that the Prandtl number is unity and that the coefficient of viscosity is proportional to the absolute temperature. It may be worth while, in this connection, to note that in reference 3, based on an analysis of flow over an impermeable wall with no heat transfer, it has been concluded that such assumptions should not lead to serious errors in the location of the separation point under ordinary conditions for air.

The analysis is first carried out in general terms, and a method for calculating the location of the separation point in a given adverse pressure gradient with a given free-stream Mach number, wall-temperature ratio, and coolant-temperature ratio is given. Since the normal-mass-flow injection distribution of the coolant, in general, must be nonuniform if the wall temperature and coolant temperature are to be uniform, it will be seen that the parameter containing the coolant temperature is a more appropriate parameter than one proportional to the magnitude of the coolant mass flow. The method presented here is a direct extension of that for an impermeable wall contained in reference 6 and, hence, possesses the same advantages of simplicity and accuracy[1] as the latter. Numerical examples are then worked out in detail for the case of a linearly decreasing velocity outside of the boundary layer, and the effect of Mach number, wall-temperature ratio, and coolant-temperature parameter (or, equivalently, magnitude of the normal mass flow of the coolant) on the separation point are thereby explicitly shown.

This work, carried out at the Polytechnic Institute of Brooklyn Aeronautical Laboratories, was sponsored by and conducted with the financial assistance of the National Advisory Committee for Aeronautics. The author hereby expresses his thanks to Mr. Richard P. Shaw and Mr. Richard G. Grape for their aid in the numerical calculations.

---

[1]Although the method presented here has been found to agree very well with presumed accurate solutions for the special case of zero heat transfer over an impermeable wall in a typical adverse pressure gradient, while all of the approximations have been made on a rational basis, this nevertheless does not constitute a rigorous proof that the method must necessarily yield equally accurate results in all other general cases. Indications, however, are that the method may be expected to be of comparable accuracy in the more general cases considered here.

SYMBOLS

A defined by equation (19c)

a_i coefficient of τ^i in velocity profile ($i = 1, \ldots, 7$)

b_1 coefficient of τ in stagnation-enthalpy profile

$c \equiv \frac{\varphi\sqrt{\lambda}}{K} \frac{T_\infty}{T_1}$ (see, also, eqs. (15) and (26))

c_p specific heat at constant pressure

c_v specific heat at constant volume

F_i defined immediately following equation (10) ($i = 1, \ldots, 5$)

F_{1s}, F_{2s} constant values of F_1 and F_2 obtained by replacing a_2 by a_{2s} and b_1 by a constant value (see, also, eqs. (28) and fig. 2)

H stagnation enthalpy, $c_pT + \frac{u^2}{2}$

h ratio of actual wall temperature to adiabatic wall temperature, T_o/T_e

K coefficient in linear viscosity-temperature relation, defined by equation (7)

k heat conductivity of fluid

l characteristic length

M Mach number

m magnitude of external velocity gradient in numerical example (see eq. (30))

Re reference Reynolds number, $\rho_\infty u_\infty l/\mu_\infty$

r,s constants defined immediately following equations (28)

S Sutherland's constant, 216° R for air

T	absolute temperature
T_c	absolute temperature of coolant
T_e	equilibrium or adiabatic wall temperature
t	variable defined by equation (8)
u,v	velocity components in x- and y- directions, respectively
x,y	distance along wall and normal to wall, respectively
α,β	constants defined by equations (21)
γ	ratio of specific heats of fluid, c_p/c_v, 1.4 for air
δ	boundary-layer thickness in xy plane
δ_t	boundary-layer thickness in xt plane

$\lambda \equiv \left(\delta_t/l\right)^2 Re$

μ	coefficient of viscosity
ρ	mass density

$\xi \equiv x/l$

$\tau \equiv t/\delta_t$

Φ	constant, $F_{1s} + F_{2s} - \beta h$ (see, also, eqs. (28) and fig. 3)
φ	normal-mass-flow parameter, $\dfrac{\rho_o v_o}{\rho_\infty u_\infty} Re^{1/2}$

Subscripts:

o	values at wall
s	values at separation point
1	values at local outer edge of boundary layer
∞	values at a reference point outside of boundary layer

A prime denotes differentiation with respect to ξ.

BASIC EQUATIONS

The partial differential equations of the compressible laminar boundary layer, for a Prandtl number of unity, can be written in the following forms:

The momentum equation:

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = \rho_1 u_1 \frac{du_1}{dx} + \frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y}\right) \tag{1}$$

The energy equation:

$$\rho u \frac{\partial H}{\partial x} + \rho v \frac{\partial H}{\partial y} = \frac{\partial}{\partial y}\left(\mu \frac{\partial H}{\partial y}\right) \tag{2}$$

The continuity equation:

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} = 0 \tag{3}$$

The quantity H is the stagnation enthalpy, defined as

$$H = c_p T + \frac{u^2}{2} \tag{4}$$

Since the pressure is assumed as constant across the boundary-layer thickness, it follows from the ideal-gas law that

$$\frac{\rho}{\rho_1} = \frac{T_1}{T} \tag{5}$$

The coefficient of viscosity μ will be assumed proportional to the absolute temperature, in the form (cf. refs. 10 and 11)[2]

$$\frac{\mu}{\mu_\infty} = K \frac{T}{T_\infty} \tag{6}$$

where K is chosen so as to satisfy exactly Sutherland's viscosity relation at the wall. If T_0 is the wall temperature, assumed in this

[2]The symbol K instead of the more familiar symbol C is used here in order to avoid any confusion with the coolant-temperature parameter C of reference 9.

analysis to be constant along the wall, then K is a constant given by

$$K = \frac{T_\infty + S}{T_0 + S}\left(\frac{T_0}{T_\infty}\right)^{1/2} \tag{7}$$

where S is a constant.

The analysis can be simplified by replacing the normal distance coordinate y by the variable t, defined by

$$y = \int_0^t \frac{T}{T_1}\,dt \tag{8}$$

By integrating equations (1) and (2) across the boundary-layer thickness with respect to y and making use of equations (3), (5), (6), and (8), the following integrodifferential equations, expressed in nondimensional form, are obtained (details are given in the appendix):

$$\frac{F_1}{2}\lambda' + \lambda\left\{F_1\frac{\rho_1'}{\rho_1} + F_1' + \frac{u_1'}{u_1}\left[F_1 + \left(1 + \frac{\gamma - 1}{2}M_1^2\right)F_2\right]\right\} =$$

$$K\frac{u_\infty}{u_1}\frac{\rho_\infty}{\rho_1}\frac{T_1}{T_\infty}\left(F_3 + \frac{\varphi}{K}\frac{T_\infty}{T_1}\sqrt{\lambda}\right) \tag{9}$$

$$\frac{F_4}{2}\lambda' + \lambda\left[F_4' + F_4\left(\frac{\rho_1'}{\rho_1} + \frac{u_1'}{u_1}\right)\right] =$$

$$K\frac{u_\infty}{u_1}\frac{\rho_\infty}{\rho_1}\frac{T_1}{T_\infty}\left[F_5 + (1 - h)\frac{\varphi}{K}\frac{T_\infty}{T_1}\sqrt{\lambda}\right] \tag{10}$$

where

$$F_1 = \int_0^1 \frac{u}{u_1}\left(1 - \frac{u}{u_1}\right) d\tau$$

$$F_2 = \int_0^1 \left[\frac{H}{H_1} - \left(\frac{u}{u_1}\right)^2\right] d\tau$$

$$F_3 = \left[\frac{\partial(u/u_1)}{\partial\tau}\right]_0$$

$$F_4 = \int_0^1 \frac{u}{u_1}\left(1 - \frac{H}{H_1}\right) d\tau$$

$$F_5 = \left[\frac{\partial(H/H_1)}{\partial\tau}\right]_0$$

The quantity h is defined here as[3]

$$h = H_0/H_1 \tag{11}$$

For a Prandtl number of 1, a well-known solution of energy equation (2) satisfying the condition of zero heat transfer at the wall is $H = \text{Constant}$. It follows that under this condition the wall temperature, called the equilibrium or adiabatic wall temperature and denoted by T_e,

[3]The quantity h here is essentially the same as that in references 7 and 9 and as G_1 in references 6 and 11.

will be

$$T_e = T_1\left(1 + \frac{\gamma - 1}{2} M_1^2\right) = T_\infty\left(1 + \frac{\gamma - 1}{2} M_\infty^2\right) \tag{12}$$

(since $H = \text{Constant}$ also along the flow at the outer edge of the boundary layer). From equations (4), (11), and (12) it follows that

$$h = T_o/T_e \tag{13}$$

Thus, at low speeds ($M_\infty \approx 0$), h denotes the ratio of the actual wall temperature to the main-stream temperature, but at high speeds the appropriate parameter h is the ratio of the actual wall temperature to the adiabatic wall temperature.

In deriving equations (9) and (10) the conditions $u = 0$ and $\rho v = \rho_o v_o$ at the wall ($y = t = 0$) and the conditions of smooth transition of the velocity and temperature profiles to their local main-stream values at the outer edge of the boundary layer ($t = \delta_t$) have been used. It should be noted, moreover, that, as in reference 11, a single boundary-layer thickness for both the velocity and the stagnation-enthalpy profiles is assumed here. Equations (9) and (10) are, in fact, a direct extension of equations (11) and (12) of reference 11 to the case of normal mass flow ($\rho_o v_o \neq 0$) at the wall. With $K = 1$, equations (9) and (10) are also equivalent to equations (8) and (9) of reference 9 if the thermal boundary-layer thickness is assumed to be the same as the dynamical boundary-layer thickness in the latter equations.

A relation between the injected normal mass flow of the cooling fluid and the desired wall and coolant temperatures can be obtained by a heat-balance equation stating that the heat transferred by the hot gas to the wall is absorbed by the coolant. This equation can be written as

$$\left(k \frac{\partial T}{\partial y}\right)_o = \rho_o v_o c_p (T_o - T_c) \tag{14}$$

Both T_o and T_c are assumed here as constant along the wall. Equation (14) implies the following nondimensional relation:

$$c \equiv \frac{\varphi}{K} \sqrt{\lambda} \frac{T_\infty}{T_1} = \frac{b_1}{h - (T_c/T_e)} \tag{15}$$

where

$$b_1 = \left[\frac{\partial(H/H_1)}{\partial\tau}\right]_o$$

The flow at the outer edge of the boundary layer, denoted by subscript 1, is assumed as known and as determined by the shape of the surface in the flow. In accordance with the isentropic-flow relations, the flow outside of the boundary layer is assumed to satisfy the relations

$$\left.\begin{aligned}
\frac{T_1}{T_\infty} &= 1 + \left(\frac{\gamma - 1}{2}\right)M_\infty^2\left[1 - \left(\frac{u_1}{u_\infty}\right)^2\right] \\
\rho_1/\rho_\infty &= (T_1/T_\infty)^{\frac{1}{\gamma-1}} \\
M_1^2 &= M_\infty^2(u_1/u_\infty)^2(T_1/T_\infty)^{-1}
\end{aligned}\right\} \tag{16}$$

Thus, it is necessary only to prescribe the velocity distribution $u_1/u_\infty(\xi)$ and the reference Mach number M_∞ outside of the boundary layer.

METHOD OF DETERMINING SEPARATION POINT

For the purpose of determining the separation point in an adverse pressure gradient, the velocity and stagnation-enthalpy profiles will be assumed, as in reference 6, as seventh-degree polynomials. The coefficients are chosen to satisfy the following boundary conditions:

At $\tau = 0$:

$$u = 0, \quad \rho v = \rho_o v_o(x), \quad T = T_o \quad (\text{or} \quad H/H_1 = h) \tag{17a}$$

$$c\,\frac{\partial(u/u_1)}{\partial\tau} = \frac{T_\infty}{T_1}\,\frac{T_o}{T_1}\,\frac{\rho_1}{\rho_\infty}\,\frac{u_1'}{u_\infty}\,\frac{\lambda}{K} + \frac{\partial^2(u/u_1)}{\partial\tau^2} \tag{17b}$$

$$h\,\frac{\partial^3(u/u_1)}{\partial\tau^3} = hc\,\frac{\partial^2(u/u_1)}{\partial\tau^2} - \frac{T_\infty}{T_1}\,\frac{T_o}{T_1}\,\frac{\rho_1}{\rho_\infty}\,\frac{u_1'}{u_\infty}\,\frac{\lambda}{K}\,\frac{\partial(H/H_1)}{\partial\tau} \tag{17c}$$

$$\frac{\partial^4(u/u_1)}{\partial\tau^4} = c\left[\frac{2}{h}\,\frac{\partial(H/H_1)}{\partial\tau} + c\right]\frac{\partial^2(u/u_1)}{\partial\tau^2} \tag{17d}$$

$$c\,\frac{\partial(H/H_1)}{\partial\tau} = \frac{\partial^2(H/H_1)}{\partial\tau^2} \tag{17e}$$

$$c\,\frac{\partial^2(H/H_1)}{\partial\tau^2} = \frac{\partial^3(H/H_1)}{\partial\tau^3} \tag{17f}$$

At $\tau = 1$:

$$u/u_1 = 1, \quad H/H_1 = 1 \tag{17g}$$

$$\frac{\partial(u/u_1)}{\partial\tau} = \frac{\partial^2(u/u_1)}{\partial\tau^2} = \frac{\partial^3(u/u_1)}{\partial\tau^3} = 0 \tag{17h}$$

$$\frac{\partial(H/H_1)}{\partial\tau} = \frac{\partial^2(H/H_1)}{\partial\tau^2} = \frac{\partial^3(H/H_1)}{\partial\tau^3} = 0 \tag{17i}$$

Equations (17b) and (17e) follow from partial differential equations (1) and (2) at the wall, while equations (17c) and (17f) can be obtained by differentiating equations (1) and (2) with respect to t and taking values at the wall. Equations (17a) to (17c) and (17e) to (17i) are essentially extensions of boundary conditions (22) to (28) of reference 11 to the case of normal mass flow at the wall (i.e., $c \neq 0$).

Equation (17d) is an additional condition introduced for the specific purpose of obtaining greater accuracy in calculating the location of the separation point in an adverse pressure gradient. This is a condition at the wall which an exact solution would necessarily satisfy at the separation point (though not necessarily elsewhere). It was originally applied in reference 12 for calculation of the separation point of incompressible flow over an impermeable wall and was subsequently applied in references 2 and 6 to compressible flow over an impermeable wall without and with heat transfer. In all of those cases the condition $\left[\partial^4(u/u_1)/\partial\tau^4\right]_0 = 0$ was used and was obtained by differentiating equation (1) twice with respect to t (or y in incompressible flow) and then taking values at the wall at the point of separation, that is, where $(\partial u/\partial t)_0 = 0$. The present condition, namely, equation (17d), is an extension of that condition for the case of a normal mass flow ($c \neq 0$) and can be obtained in a manner essentially similar to that shown in detail, for an impermeable wall, in reference 6.

The seventh-degree stagnation-enthalpy and velocity profiles satisfying boundary conditions (17a) to (17i) can be written in the following dimensionless forms:

$$\frac{H}{H_1} = h + (1 - h)\left(35\tau^4 - 84\tau^5 + 70\tau^6 - 20\tau^7\right) +$$

$$b_1\left[\left(\tau - 20\tau^4 + 45\tau^5 - 36\tau^6 + 10\tau^7\right) +\right.$$

$$\frac{c}{2}\left(\tau^2 - 10\tau^4 + 20\tau^5 - 15\tau^6 + 4\tau^7\right) +$$

$$\left.\frac{1}{6}c^2\left(\tau^3 - 4\tau^4 + 6\tau^5 - 4\tau^6 + \tau^7\right)\right] \tag{18}$$

$$\frac{u}{u_1} = \sum_{i=1}^{7} a_i\tau^i \tag{19a}$$

where

$$
\left.
\begin{aligned}
a_1 &= \left(A + 2a_2\right)/c \\
a_2 &= \frac{(7/2)c - 2A\left(1 - \frac{b_1 c}{30h}\right)}{4 + c + \frac{c^2}{120}\left(16 + c + \frac{2b_1}{h}\right)} \\
a_3 &= \frac{ca_2}{3} - \frac{b_1 A}{6h} \\
a_4 &= \frac{ca_2}{12}\left(\frac{2b_1}{h} + c\right) \\
a_5 &= -\frac{21}{4} + \frac{b_1 A}{2h} - a_2\left[\frac{5}{2} + c + \frac{3}{16}\, c\left(\frac{2b_1}{h} + c\right)\right] \\
a_6 &= 7 - \frac{8}{15}\,\frac{b_1 A}{h} + a_2\left[3 + \frac{16}{15}\, c + \frac{3}{20}\, c\left(\frac{2}{h}\, b_1 + c\right)\right] \\
a_7 &= -\frac{5}{2} + \frac{b_1 A}{6h} - a_2\left[1 + \frac{c}{3} + \frac{c}{24}\left(\frac{2b_1}{h} + c\right)\right]
\end{aligned}
\right\} \tag{19b}
$$

and

$$
A = h\left(1 + \frac{\gamma - 1}{2} M_1{}^2\right)\frac{u_1{}'}{u_\infty}\,\frac{\lambda}{K}\left(\frac{T_1}{T_\infty}\right)^{\frac{2-\gamma}{\gamma-1}} \tag{19c}
$$

By substituting expressions (18) and (19a) to (19c) for the stagnation-enthalpy and velocity profiles into equations (9) and (10), the latter become ordinary differential equations for $\lambda(\xi)$ and $b_1(\xi)$. Proceeding in a manner analogous to that in references 9 and 11, one can obtain a general approximate solution of equation (9) by making the simplifying approximation that in the expressions for F_1 and F_2 (and only there) the $a_2(\xi)$ terms may be replaced by constant values for the entire flow. Moreover, $b_1(\xi)$ is assumed to be replaceable by a constant value in equation (9). Hence, in accordance with equation (15), c will be a constant, since h and T_c (and hence T_c/T_e) are assumed here as uniform along the wall. By making use of equation (19a) in conjunction with the expression for a_1 in equations (19b) and (19c), equation (9) can thus be written as the following linear first-order differential equation for $\lambda(\xi)$:

$$\frac{F_{1s}}{2}\lambda' + \lambda\left\{F_{1s}\frac{\rho_1'}{\rho_1} + \frac{u_1'}{u_1}\left[\Phi + \left(\frac{\gamma - 1}{2}\right)M_1^2(\Phi - F_{1s})\right]\right\} =$$

$$K\frac{u_\infty}{u_1}\frac{\rho_\infty}{\rho_1}\frac{T_1}{T_\infty}(c + \alpha) \tag{20}$$

where F_{1s} and F_{2s} are the constants obtained by giving a_2 and b_1 constant values in F_1 and F_2, respectively, while Φ is a constant defined by

$$\Phi = F_{1s} + F_{2s} - \beta h$$

Moreover, α and β are constants defined by the relation $a_1 = \alpha + \beta A$. Thus

$$\left.\begin{aligned}
\alpha &= \frac{7}{4 + c + \frac{c^2}{120}\left(16 + c + \frac{2b_1}{h}\right)} \\
\beta h &= \frac{h + \frac{c}{120}\left[(16 + c)h + 2b_1\right] + \frac{2}{15}b_1}{4 + c + \frac{c^2}{120}\left(16 + c + \frac{2b_1}{h}\right)}
\end{aligned}\right\} \tag{21}$$

By the use of relations (16), the following solution of equation (20), with the condition $\lambda = 0$ at $\xi = 0$, is obtained:

$$\lambda(\xi) = \frac{2}{F_{1s}}(c + \alpha)K \frac{\int_0^{\xi} \left(\frac{u_1}{u_\infty}\right)^{\frac{2}{F_{1s}}\Phi - 1} \left(\frac{T_1}{T_\infty}\right)^{\frac{2\gamma-1}{\gamma-1} - \frac{\Phi}{F_{1s}}} d\xi}{\left(\frac{u_1}{u_\infty}\right)^{\frac{2}{F_{1s}}\Phi} \left(\frac{T_1}{T_\infty}\right)^{\frac{\gamma+1}{\gamma-1} - \frac{\Phi}{F_{1s}}}} \tag{22}$$

It may be noted that this expression for the dimensionless squared boundary-layer thickness $\lambda(\xi)$ in the xt plane is of a form quite similar to that (cf. refs. 11 and 6) for zero normal mass flow ($c = 0$). In fact, as has been seen, the introduction of a small[4] normal mass flow of cooling air at the wall does not introduce any additional difficulties into the present mathematical analysis, and no simplifying approximations in addition to those made in references 11 and 6 for an impermeable wall need be made here. This is due essentially to the fact that the mass-flow parameter φ appears only in the form defined by c and that according to heat-balance equation (15) with uniform wall and coolant temperatures c can be approximated well by a constant.

Separation of the flow will occur where $(\partial u/\partial y)_0 = 0$ and, hence, where $a_1 = 0$. According to equations (19b) and (19c), this implies that the value of λ (to be denoted as λ_s) required for separation will be:

$$\lambda_s = -7K \frac{\left(\frac{T_1}{T_\infty}\right)^{-\frac{2-\gamma}{\gamma-1}}}{\left(\frac{u_1'}{u_\infty}\right)\left(1 + \frac{\gamma - 1}{2} M_1^2\right)\left\{h + \frac{2}{15} b_1 + \frac{c}{120}\left[(16 + c)h + 2b_1\right]\right\}} \tag{23}$$

The value (assumed constant) of b_1 now remains to be determined and, thence, the value of the coolant parameter c. The value of b_1 can be obtained in a manner analogous to that developed in reference 11 for an impermeable wall. However, for flows which, at the leading edge, behave like those in a zero pressure gradient, that is, flows for which u_1 is finite while $\lambda = 0$ at $\xi = 0$, a good approximation for b_1 can be obtained by calculating the value corresponding to the flow over a flat plate (cf. ref. 6).

---

[4]That is, sufficiently small that the boundary-layer approximations of Prandtl remain valid.

To determine the value of b_1 for flow over a flat plate ($u_1' = 0$) it is first noted that for this case, with the wall temperature uniform, equations (1) and (2) imply a linear relation between the stagnation-enthalpy and the velocity profiles:

$$\frac{H}{H_1} = h + (1 - h)\left(\frac{u}{u_1}\right) \tag{24}$$

Equation (24) is valid regardless of whether or not a normal mass flow exists at the wall. If it is assumed (as, e.g., in ref. 11) that the velocity profiles in the boundary layer over a flat plate can, in general, be well represented by sixth-degree polynomials,[5] then it follows from equation (24) that the stagnation-enthalpy profiles must in this case also be sixth-degree polynomials. Consequently, an accurate value of b_1 for the flow over a flat plate can be obtained simply by setting the coefficient of τ^7 in equation (18) equal to zero. This yields

$$b_1 = \frac{120(1 - h)}{60 + 12c + c^2} \tag{25}$$

In analogy to the procedure in reference 6, this is the value of b_1 which will be assumed herein for the purpose of calculating the separation point. Substitution into equation (15) for b_1 according to equation (25) yields

$$c(60 + 12c + c^2) = 120\,\frac{1 - h}{h - (T_c/T_e)} \tag{26}$$

Thus, for a given (positive) value of the parameter $(1 - h)/\left[h - (T_c/T_e)\right]$, the value of c can be directly determined by calculating the real positive root of cubic equation (26). Although the basic coolant-temperature parameter is $(1 - h)/\left[h - (T_c/T_e)\right]$, it is seen that it is more convenient to treat c as the coolant-temperature parameter. The parameter c as a function of $(1 - h)/\left[h - (T_c/T_e)\right]$ is shown in figure 1.

The values of the constants F_{1s} and F_{2s} and, hence, Φ can be obtained by inserting expressions (18) and (19a) to (19c) for the stagnation-enthalpy and velocity profiles into the integral expressions

---

[5]These profiles would be chosen to satisfy all of boundary conditions (17a) to (17i) with the exception of condition (17d).

for F_1 and F_2 given immediately following equations (9) and (10). As in references 2 and 6, the constant value of a_2 (to be denoted by a_{2s}) will be chosen here as the value of $a_2(\xi)$ at the separation point. Setting $a_1 = 0$ and using equations (19b) for a_1 and a_2, in addition to equation (19c), it is found that

$$a_{2s} = \frac{7/2}{1 + \frac{2}{15}\frac{b_1}{h} + \frac{c}{120}\left(16 + c + \frac{2b_1}{h}\right)} \tag{27}$$

The explicit expressions for the constants F_{1s} and Φ are thus found to be

$$\left.\begin{aligned}
F_{1s} = {} & -0.273310 + 0.370581a_{2s} - 0.074867a_{2s}^{2} + 0.002863a_{2s}r - \\
& 0.001183a_{2s}^{2}r - 0.000004727a_{2s}^{2}r^{2} - 0.023592(s - 2a_{2s}c) + \\
& 0.009678a_{2s}(s - 2a_{2s}c) + 0.00007701a_{2}r(s - 2a_{2s}c) - \\
& 0.0003141(s - 2a_{2s}c)^{2} \\
\Phi = {} & 2F_{1s} + \frac{11}{16} + \frac{h}{2} + b_1\left(\frac{3}{28} + \frac{c}{84} + \frac{c^2}{1,680}\right) - \\
& a_{2s}\left(\frac{37}{168} + \frac{23}{840}c\right) + \frac{1}{6,720}(92s - 11a_{2s}r) - \beta h
\end{aligned}\right\} \tag{28}$$

where

$$r \equiv \frac{c}{h}(2b_1 + ch)$$

$$s \equiv -2a_{2s}b_1/h$$

and b_1, a_{2s}, and β are given respectively by equations (25), (27), and (21). Thus F_{1s} and Φ are functions only of the temperature ratio h and the coolant parameter c. These functions are shown in figures 2 and 3, which may be used to facilitate actual calculations.[6]

In order to determine the separation point over a sweat-cooled wall in a given adverse pressure gradient ($u_1' < 0$) with a given reference Mach number M_∞, given uniform wall-temperature ratio h, and given uniform coolant-temperature ratio T_o/T_e, it is necessary only to calculate $\lambda(\xi)$ in accordance with equation (22) and to determine the value of ξ at which $\lambda(\xi) = \lambda_s$, where λ_s is given by equation (23). This method includes the case of flow over an impermeable wall (zero normal mass flow at the wall), for which $c = 0$ and h is considered as arbitrary. The coolant normal-mass-flow distribution, as given by φ and implicit in this analysis, can be obtained from equation (15) after $\lambda(\xi)$ has been determined. Thus,

$$\varphi(\xi) = K \frac{T_1}{T_\infty} \frac{c}{\sqrt{\lambda}} \tag{29}$$

where c is determined by equation (26) (or fig. 1).

NUMERICAL EXAMPLE

The implications of this analysis for the effects of Mach number, and simultaneous cooling of the wall and normal mass flow, on laminar separation over a sweat-cooled wall can best be shown by a numerical example. For this purpose, the case of a linearly diminishing velocity outside of the boundary layer will be treated in detail. Thus, it will be assumed that

$$u_1/u_\infty = 1 - m\xi \tag{30}$$

where m is a positive constant. For the case represented by equation (30), the separation point will be calculated as a function of the coolant-temperature ratio (and thus implicitly as a function of the

---

[6] If F_{1s} and Φ are calculated directly from equations (28) instead of read off from figures 2 and 3, care must be taken to use a sufficient number of significant figures, since relatively small differences may occur here.

required magnitude and distribution of the normal mass flow of the coolant) for various fixed values of the wall-temperature ratio T_o/T_e.

For $M_\infty = 0$, that is, for low-speed (but nevertheless compressible) flows, equations (16) imply $T_1/T_\infty = 1$, and equations (22) and (23) in conjunction with equation (30) then yield the following relatively simple expression for the separation point as a function of the wall-temperature ratio h and the coolant-temperature parameter c:

$$m\xi_s = 1 - \left(1 + \frac{7\Phi}{(c + \alpha)\left\{h + \frac{2}{15} b_1 + \frac{c}{120}\left[(16 + c)h + 2b_1\right]\right\}}\right)^{-\frac{F_{1s}}{2\Phi}} \tag{31}$$

where α, b_1, F_{1s}, and Φ are given by equations (21), (25), (27), and (28) (or figs. 2 and 3 instead of equations (27) and (28)).

The results represented by equation (31) are shown explicitly in figure 4, where $m\xi_s$ is shown as a function of the coolant-temperature ratio T_c/T_e for various fixed values of the wall-temperature ratio T_o/T_e. Thus, figure 4 shows essentially the separation point as a function of the implicitly required normal mass flow for various fixed degrees of cooling of the wall. Along any curve here for a given value of h, the higher the coolant-temperature ratio T_c/T_e, the greater will be the magnitude of the corresponding normal-mass-flow parameter $\varphi(\xi)$. This is illustrated in figure 5, based on equation (29), which shows the required normal-mass-flow distribution for two different values of T_c/T_e with $h = 0.6$.

The case $T_c/T_e = 0$ in figure 4 represents the limiting case in which the required normal mass flow of the coolant is a minimum for the given desired wall-temperature ratio $h(\equiv T_o/T_e)$, while the case $T_c/T_e = h$ represents the opposite limiting case of an indefinitely large required magnitude of coolant mass flow.

Figure 4 shows vividly the influence of a normal mass flow in promoting separation. This can be seen from the fact that for a given wall-temperature ratio h the separation point moves upstream as the coolant-temperature ratio is increased. It is further significant to to note that all of the curves for $h < 1$ are below the straight line corresponding to $h = 1$. The latter line represents the case of an adiabatic impermeable wall. This shows that the net effect of simultaneous normal mass flow and cooling of the wall in flow over a transpiration-cooled wall is, in all cases, to move the separation point upstream.

Thus the unfavorable effect of normal mass flow, as such, on separation evidently predominates, in this example, over the favorable effect (see, e.g., ref. 6) of cooling of the wall, as such. This predominating influence of the normal mass flow can be further seen in figure 4 by noting that for a given coolant temperature the cooler it is desired to maintain the sweat-cooled wall, the sooner laminar separation will occur. Such is the case even if the minimum required normal mass flow of the coolant is used (i.e., $T_c = 0$) for each desired wall temperature.[7]

To show the influence of Mach number, the separation point as a function of the wall temperature and coolant temperature was calculated for $M_\infty = 3$, on the basis of equations (22) and (23). The integration indicated in equation (22) was carried out numerically by Simpson's rule. The results of this calculation are shown in figure 6. These results are seen to be quite similar to those for zero Mach number (fig. 4) and, hence, the same conclusions regarding the effect of simultaneous cooling of the wall and normal mass flow which were drawn on the basis of low-speed flow ($M_\infty = 0$) remain valid for higher speeds, at least for $M_\infty = 3$. Separation, however, is seen to occur earlier, that is, farther upstream, at the high Mach number, for the same pairs of values of T_o/T_e and T_c/T_e.

CONCLUSIONS

From an analysis of separation in compressible flow over a transpiration-cooled wall maintained at a uniform wall temperature by an appropriate distribution of normal mass flow, the following conclusions, based on a Prandtl number of unity and a linear viscosity-temperature relation, can be drawn:

1. For flow in a given adverse pressure gradient and at a given reference Mach number, the separation point as a function of the wall temperature and the coolant temperature can be calculated in a simple manner by the approximate method developed here. The pertinent temperature parameters at high speeds are h and $(1 - h)/\left[h - (T_c/T_e)\right]$, where $h = T_o/T_e$, T_o is the wall temperature, T_c is the coolant temperature, and T_e is the adiabatic wall temperature.

---

[7]The results in figure 4 although qualitatively similar to those in reference 7 differ from the latter in that the corresponding curves in reference 7 intersected one another, while there were pairs of values of T_o/T_e and T_c/T_e for which $m\xi_s$ was downstream of that corresponding to an adiabatic impermeable wall. This was due to the use of fourth-degree velocity profiles in reference 7.

2. From the numerical example calculated, it is seen that the net effect of simultaneous cooling of the wall and normal mass flow is to move the separation point upstream. Thus, the unfavorable effect of normal mass flow, as such, on separation predominates over the favorable effect of cooling of the wall, as such. This conclusion is valid at supersonic speeds as well as at low speeds.

3. From the same numerical example, it is found that for fixed pairs of values of T_o/T_e and T_c/T_e and a fixed external flow distribution the effect of a higher Mach number is to move the separation point upstream.

Polytechnic Institute of Brooklyn,
Brooklyn, N. Y., December 6, 1954.

APPENDIX

DERIVATION OF INTEGRAL EQUATIONS (9) AND (10)

Integrating equation (1) with respect to y over the boundary-layer thickness δ yields

$$\int_0^\delta \rho u \frac{\partial u}{\partial x} dy + \int_0^\delta \rho v \frac{\partial u}{\partial y} dy - \int_0^\delta \rho_1 u_1 u_1' dy = -\left(\mu \frac{\partial u}{\partial y}\right)_o \tag{A1}$$

where a prime here denotes differentiation with respect to x. From continuity equation (3) it follows that

$$\frac{\partial(\rho v)}{\partial y} = -\frac{\partial(\rho u)}{\partial x} \tag{A2}$$

and

$$(\rho v)_{y=\delta} = \rho_o v_o - \int_0^\delta \frac{\partial(\rho u)}{\partial x} dy \tag{A3}$$

Integrating the second term of equation (A1) by parts and making use of equations (A2) and (A3), it is found that equation (A1) can be written, with a suitable rearrangement and combination of terms, in the form

$$\int_0^\delta \frac{\partial}{\partial x} (\rho u^2 - \rho u u_1) dy + \rho_o v_o u_1 + \int_0^\delta (\rho u u_1' - \rho_1 u_1 u_1') dy = -\left(\mu \frac{\partial u}{\partial y}\right)_o \tag{A4}$$

Applying the theorem on differentiation of an integral with respect to a parameter α, namely,

$$\frac{d}{d\alpha} \int_{a(\alpha)}^{b(\alpha)} f(y,\alpha) dy = \int_{a(\alpha)}^{b(\alpha)} \frac{\partial f(y,\alpha)}{\partial \alpha} dy - f(a,\alpha) \frac{da}{d\alpha} + f(b,\alpha) \frac{db}{d\alpha}$$

equation (A4) becomes

$$\frac{d}{dx}\int_0^\delta \rho u\left(u_1 - u\right) dy + u_1' \int_0^\delta \left(\rho_1 u_1 - \rho u\right) dy = \rho_o v_o u_1 + \left(\mu \frac{\partial u}{\partial y}\right)_o \tag{A5}$$

From equation (8) it follows that

$$dy = \left(T/T_1\right) dt = \left(\rho_1/\rho\right) dt \tag{A6}$$

Moreover, $\partial u/\partial y = (\partial u/\partial t)\ (dt/dy) = (T_1/T)\ (\partial u/\partial t)$. Thus, transforming equation (A5) from the xy plane to the xt plane and introducing $\tau \equiv t/\delta_t$ (with $t = \delta_t$ when $y = \delta$), equation (A5) becomes

$$\frac{d}{dx}\left(F_1 \rho_1 u_1^2 \delta_t\right) + \rho_1 u_1 u_1' \delta_t \int_0^1 \left(\frac{T}{T_1} - \frac{u}{u_1}\right) d\tau = \rho_o v_o u_1 + \mu_\infty K \frac{T_1}{T_\infty}\frac{u_1}{\delta_t} F_3 \tag{A7}$$

where F_1 and F_3 are defined as in the main text (following eq. (10)), while use has been made of the viscosity-temperature relation (6). Equation (A7) can be written in the following nondimensional form:

$$\left(\frac{\delta_t}{l}\right)\left[F_1 \frac{\rho_1}{\rho_\infty}\left(\frac{u_1}{u_\infty}\right)^2\left(\frac{\delta_t}{l}\right)\right]' + \left(\frac{\rho_1}{\rho_\infty}\right)\left(\frac{u_1}{u_\infty}\right)\left(\frac{u_1}{u_\infty}\right)'\left(\frac{\delta_t}{l}\right)^2 \int_0^1 \left(\frac{T}{T_1} - \frac{u}{u_1}\right) d\tau =$$

$$\left(\frac{\rho_o v_o}{\rho_\infty u_\infty}\right)\left(\frac{u_1}{u_\infty}\right)\left(\frac{\delta_t}{l}\right) + \frac{\mu_\infty}{\rho_\infty u_\infty l} K \frac{T_1}{T_\infty}\frac{u_1}{u_\infty} F_3 \tag{A8}$$

where now the prime indicates differentiation with respect to ξ, $\xi \equiv x/l$, and l is a characteristic length. Carrying out straightforwardly the indicated differentiation of the first term in equation (A8) and thereby obtaining four terms, combining these individual terms suitably with the other terms in equation (A8), and dividing through by $\left(\rho_1/\rho_\infty\right)\left(u_1/u_\infty\right)^2\left(\mu_\infty/\rho_\infty u_\infty l\right)$ yield

$$\frac{F_1}{2}\lambda' + \lambda\left\{F_1\frac{\rho_1'}{\rho_1} + F_1' + \frac{u_1'}{u_1}\left[2F_1 + \int_0^1\left(\frac{T}{T_1} - \frac{u}{u_1}\right)d\tau\right]\right\} =$$

$$K\frac{u_\infty}{u_1}\frac{\rho_\infty}{\rho_1}\frac{T_1}{T_\infty}\left(F_3 + \frac{\varphi}{K}\frac{T_\infty}{T_1}\sqrt{\lambda}\right) \tag{A9}$$

where $\lambda \equiv \left(\delta_t/l\right)^2 \mathrm{Re}$ and $\varphi = \frac{\rho_o v_o}{\rho_\infty u_\infty}\mathrm{Re}^{1/2}$. Observing that from the definition of the stagnation enthalpy H

$$\frac{T}{T_1} = \frac{H}{H_1}\left(1 + \frac{\gamma - 1}{2}M_1^2\right) - \frac{\gamma - 1}{2}M_1^2\left(\frac{u}{u_1}\right)^2 \tag{A10}$$

and noting the definition of F_1, it can be seen that the bracketed factor of u_1'/u_1 in equation (A9) is the same as that in equation (9). Thus, equation (A9) is identical with equation (9) of the main text.

Equation (10) of the main text can be quite similarly derived from partial differential equation (2).

REFERENCES

1. Duwez, Pol, and Wheeler, H. L., Jr.: Experimental Study of Cooling by Injection of a Fluid Through a Porous Material. Jour. Aero. Sci., vol. 15, no. 9, Sept. 1948, pp. 509-521.

2. Morduchow, Morris, and Clarke, Joseph H.: Method for Calculation of Compressible Laminar Boundary-Layer Characteristics in Axial Pressure Gradient With Zero Heat Transfer. NACA TN 2784, 1952.

3. Gadd, G. E.: Some Aspects of Laminar Boundary Layer Separation in Compressible Flow With No Heat Transfer to the Wall. Aero. Quart., vol. 4, pt. 2, Feb. 1953, pp. 123-150.

4. Illingworth, C. R.: The Effect of Heat Transfer on the Separation of a Compressible Laminar Boundary Layer. Quart. Jour. Mech. and Appl. Math., vol. 7, pt. 1, Mar. 1954, pp. 8-34.

5. Tani, Itiro: On the Approximate Solution of the Laminar Boundary-Layer Equations. Jour. Aero. Sci., vol. 21, no. 7, July 1954, pp. 487-495, 504.

6. Morduchow, Morris, and Grape, Richard G.: Separation, Stability, and Other Properties of Compressible Laminar Boundary Layer With Pressure Gradient and Heat Transfer. NACA TN 3296, 1955.

7. Morduchow, Morris, and Galowin, Lawrence: The Compressible Laminar Boundary Layer in a Pressure Gradient Over a Surface Cooled by Fluid Injection. Proc. First Iowa Thermodynamics Symposium, State Univ. of Iowa, Apr. 27-28, 1953, State Univ. of Iowa (Iowa City), pp. 143-169.

8. Stewartson, K.: Correlated Incompressible and Compressible Boundary Layers. Proc. Roy. Soc. (London), ser. A, vol. 200, no. 1060, Dec. 22, 1949, pp. 84-100.

9. Morduchow, Morris: On Heat Transfer Over a Sweat-Cooled Surface in Laminar Compressible Flow With a Pressure Gradient. Jour. Aero. Sci., vol. 19, no. 10, Oct. 1952, pp. 705-712.

10. Chapman, D. R., and Rubesin, M. W.: Temperature and Velocity Profiles in the Compressible Laminar Boundary Layer With Arbitrary Distribution of Surface Temperature. Jour. Aero. Sci., vol. 16, no. 9, Sept. 1949, pp. 547-565.

11. Libby, Paul A., and Morduchow, Morris: Method for Calculation of Compressible Laminar Boundary Layer With Axial Pressure Gradient and Heat Transfer. NACA TN 3157, 1954.

12. Timman, R.: A One Parameter Method for the Calculation of Laminar Boundary Layers. Rep. F. 35, Nationaal Luchtvaartlaboratorium, 1949, pp. F 29 - F 46.

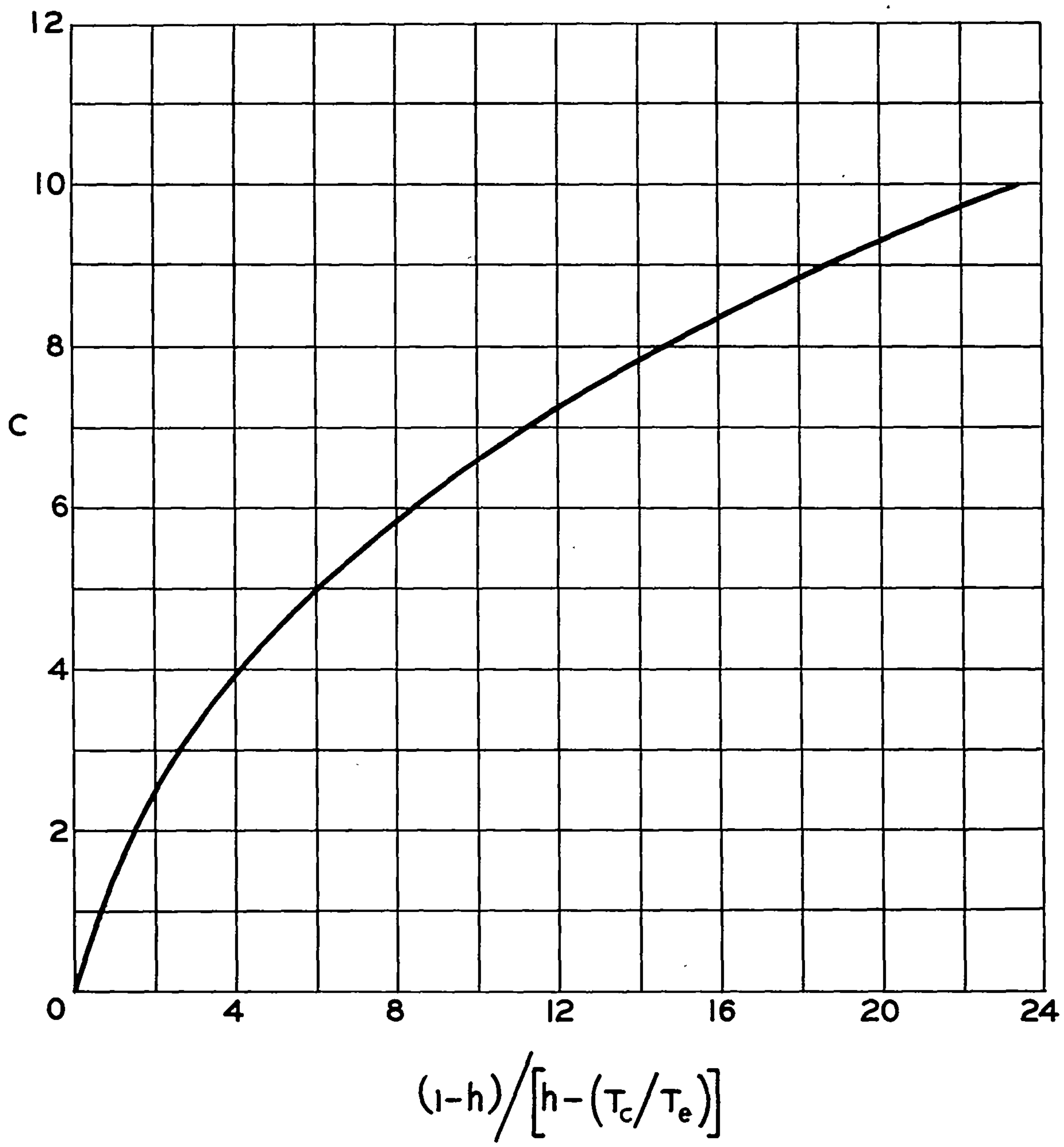

Figure 1.- Coolant parameter c as a function of $(1 - h)\Big/\left[h - (T_c/T_e)\right]$ (eq. (26)).

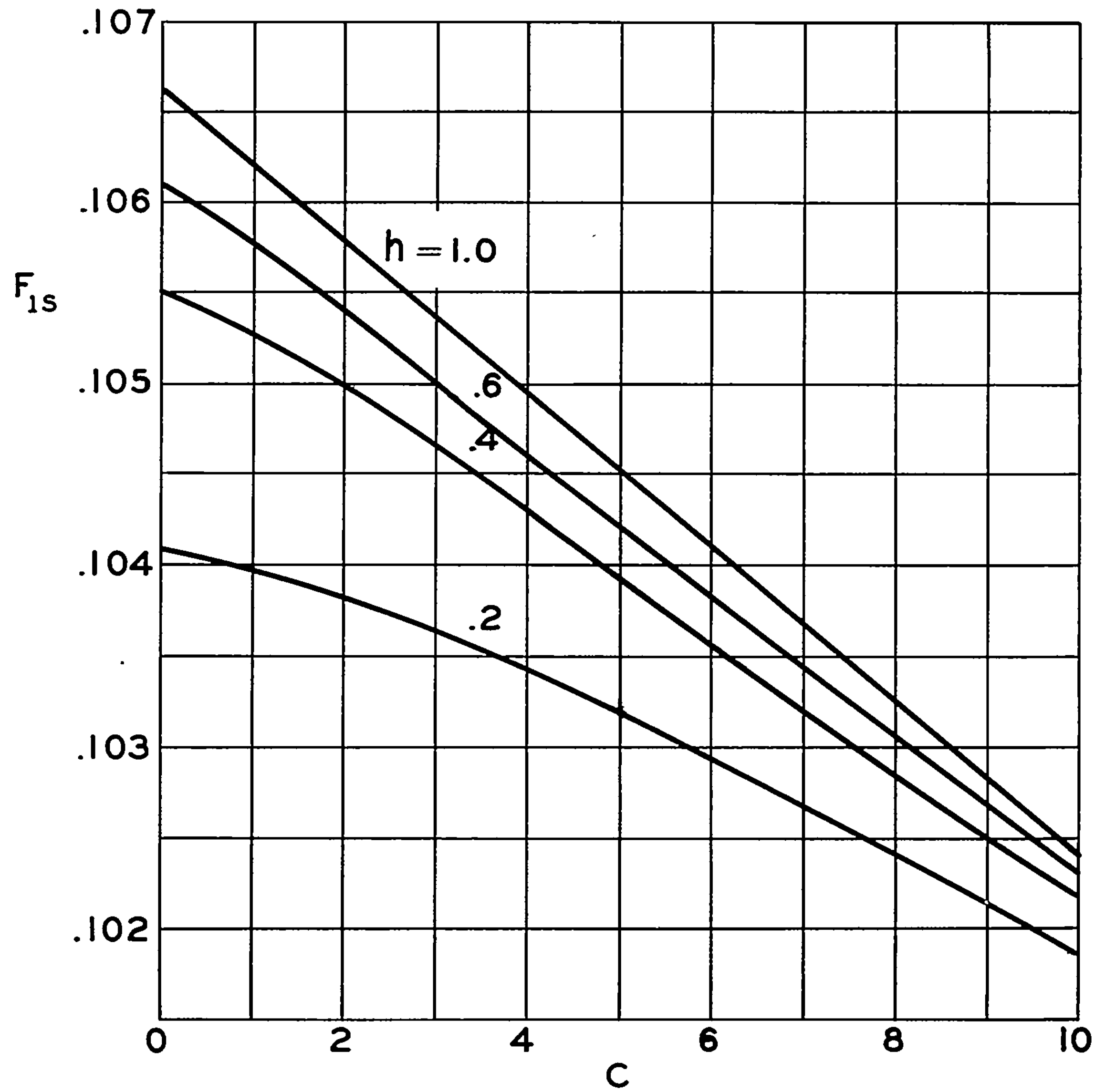

Figure 2.- Constant F_{1s} as a function of temperature ratio h and coolant parameter c.

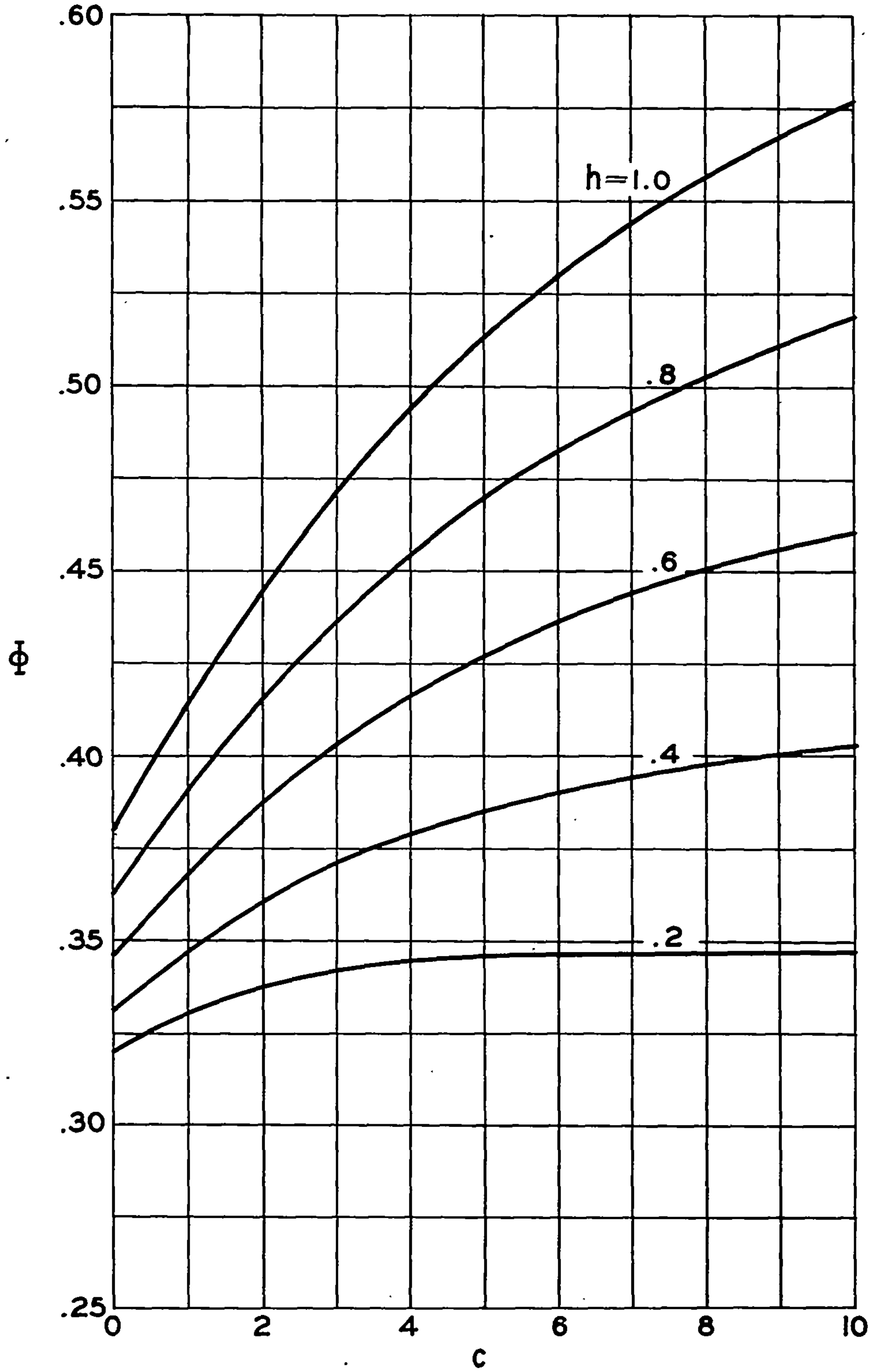

Figure 3.- Constant Φ as a function of temperature ratio h and coolant parameter c.

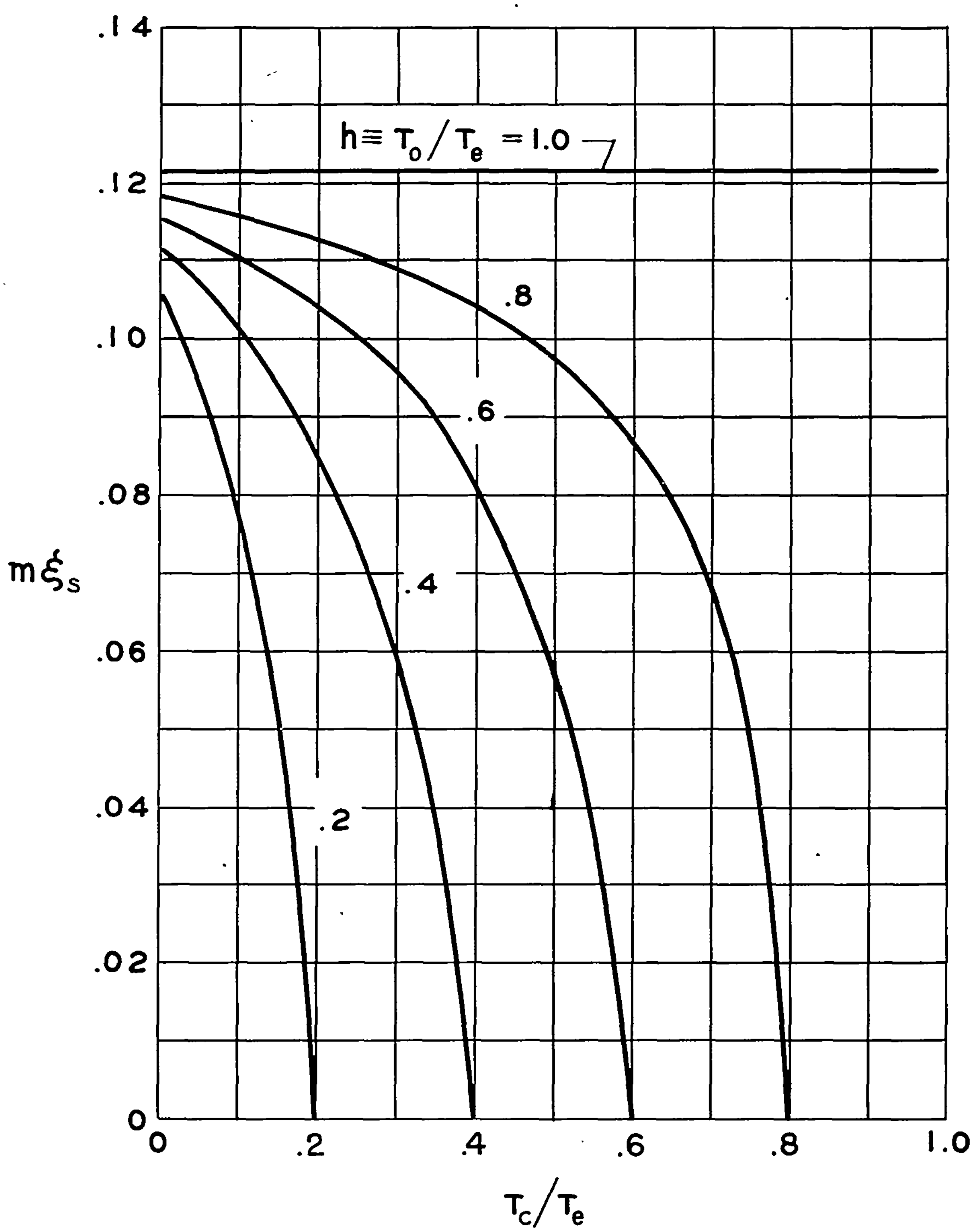

Figure 4.- Separation point as a function of wall temperature and coolant temperature. $u_1/u_\infty = 1 - m\xi$; $M_\infty = 0$.

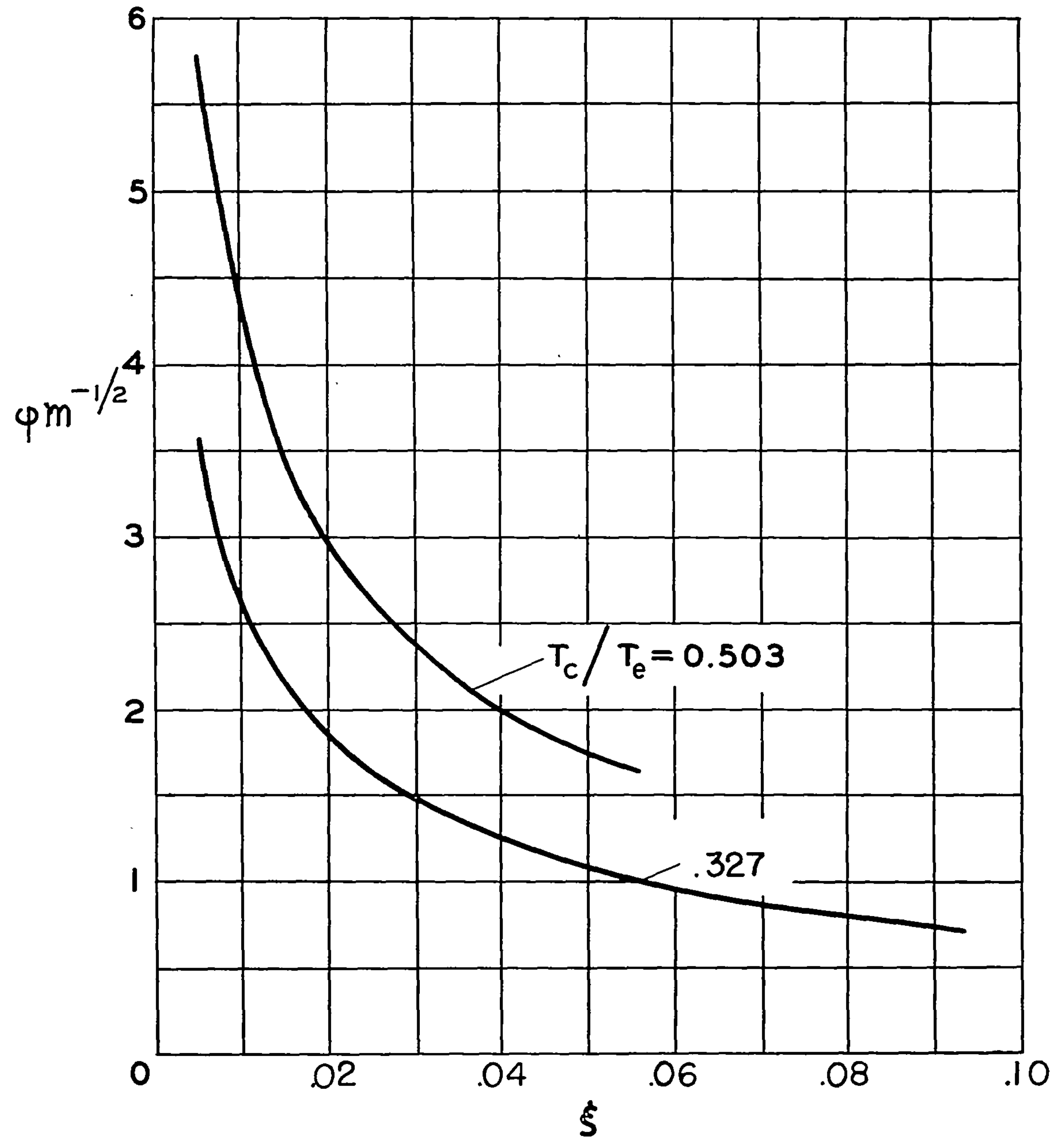

Figure 5.- Required normal-mass-flow distribution of coolant to maintain wall-temperature ratio of $h = 0.6$. $u_1/u_\infty = 1 - m\xi$; $M_\infty = 0$; $S/T_\infty = 0.416$.

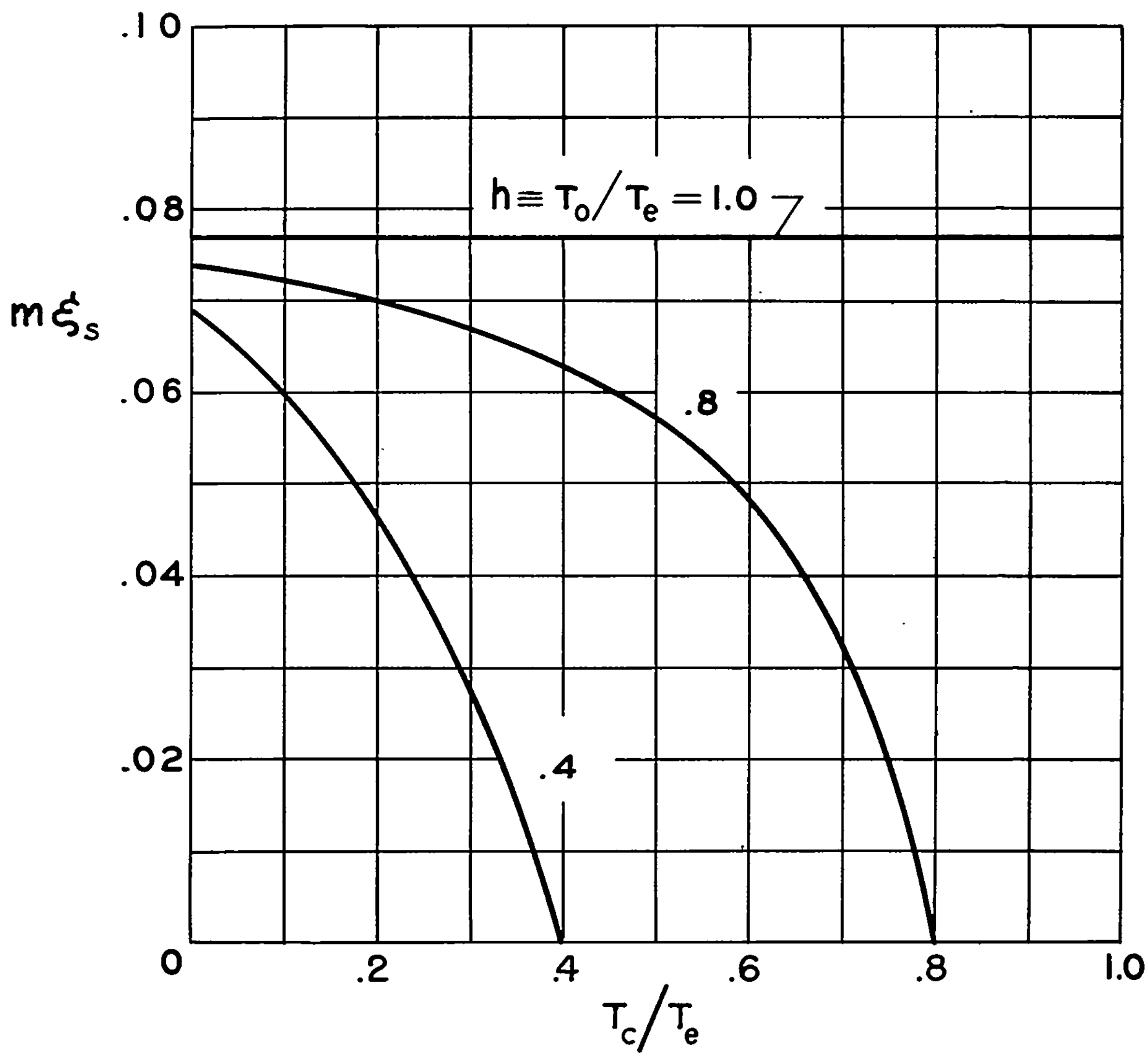

Figure 6.- Separation point as a function of wall temperature and coolant temperature. $u_1/u_\infty = 1 - m\xi$; $M_\infty = 3$.

NACA - Langley Field, Va.